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A simple spectral condition implying separability for states of bipartite quantum systems

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Abstract
We give a simple spectral condition in terms of the ordered eigenvalues of the state of a bipartite quantum system which is sufficient for separability.

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We consider quantum systems where the underlying Hilbert space $\mathcal{H}$ is the tensor product of two finite-dimensional Hilbert spaces. A state of such a system is identified with a density operator and is said to be separable if it can be written as a convex sum of pure product states of the system, that is to say, vector states where the vectors are product vectors. The separable states form a convex subset of the states of the system.

For the simplest bipartite composite system, we have the following result.

Theorem 1. *If the eigenvalues* $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ *of the two-qubit state* ρ *satisfy* $3\lambda_1 + \sqrt{2}\lambda_2 + (3 - \sqrt{2})\lambda_3 \leqslant 2$, *then* ρ *is separable.*

The states satisfying the inequality have spectra in the simplex spanned by the spectra (always written taking into account multiplicities and nonincreasingly) $(1/2, 1/6, 1/6, 1/6)$, $((2+\sqrt{2})/8, (2+\sqrt{2})/8, (2-\sqrt{2})/8, (2-\sqrt{2})/8)$, $(1/3, 1/3, 1/3, 0)$ and $(1/4, 1/4, 1/4, 1/4)$.

The method used to prove the above result also gives a different proof of the following result given in ([1], theorem 3).

Theorem 2. *If the eigenvalues* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d$ *of the state* ρ *of a bipartite quantum system of dimension* d *satisfy* $3\lambda_d + (d-1)\lambda_{d-1} \geqslant 1$, *then* ρ *is separable.*

Both results provide simple spectral criteria ensuring separability. In the case of two qubits ($d = 4$), theorem 2 is much weaker than theorem 1.

Before proceeding to the proofs, we compare theorem 1 with other available results of the same nature, that is, conditions on the spectrum implying separability of the state. Given a state ρ of a d-dimensional bipartite quantum system, we let $\mathrm{spec}(\rho) = (\rho_1, \rho_2, \ldots, \rho_d)$

denote the vector of repeated eigenvalues of ρ enumerated so that $\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_d$. In the two-qubit case, let $\Sigma = \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4) : \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1\}$ be all possible state spectra. Theorem 1 asserts that if spec(ρ) lies in $\mathcal{A} := \{\lambda \in \Sigma : 3\lambda_1 + \sqrt{2}\lambda_2 + (3 - \sqrt{2})\lambda_3 \leqslant 2\}$, then ρ is separable. One of the first and most useful results of this nature is that of [2]: if tr$(\rho^2) \leqslant 1/3$, then ρ is separable. In terms of spectra, this is as follows: spec$(\rho) \in \mathcal{B} := \left\{\lambda \in \Sigma : \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \leqslant 1/3\right\}$ implies ρ is separable. Although $\mathcal{A} \cap \mathcal{B}$ is quite large, $\mathcal{A} \neq \mathcal{B}$, and the conditions defining $\mathcal{A}$ and $\mathcal{B}$ capture different (convex) sets of separable states. To see this, observe that $\lambda \equiv ((2+\sqrt{2})/8, (2+\sqrt{2})/8, (2-\sqrt{2})/8, (2-\sqrt{2})/8) \in \mathcal{A}$ but $\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2 = 3/8 > 1/3$. Moreover, $(\sqrt{2}/3)(3/4, 1/4, 0, 0) + (1 - (\sqrt{2}/3))(1/4, 1/4, 1/4, 1/4)$ lies in $\mathcal{B}$ but not in $\mathcal{A}$. The determination of 'maximally entangled' states of two qubits by Verstraete, Audenaert and De Moor [3] has the following as a consequence[1]: let $\mathcal{C} := \{\lambda \in \Sigma : \lambda_1 - \lambda_3 - 2\sqrt{\lambda_2\lambda_4} \leqslant 0\}$; then ρ is separable if spec$(\rho) \in \mathcal{C}$. The inequality $\sqrt{(ta + (1-t)b)(tc + (1-t)d)} \geqslant t\sqrt{ac} + (1-t)\sqrt{bd}$ valid for $0 \leqslant t \leqslant 1$ and $a, b, c, d \geqslant 0$ shows immediately that $\mathcal{C}$ is convex. By the results of [3], one has $\mathcal{B} \subset \mathcal{C}$.[2] One verifies that the four vertices of $\mathcal{A}$ given after the statement of theorem 1 lie in $\mathcal{C}$ so that $\mathcal{A} \subset \mathcal{C}$ because $\mathcal{A}$ is the convex hull of its four vertices.

The proof of the two stated results uses certain tools developed in [1] which we briefly present. Consider the maximally mixed state $\tau = \mathbf{1}/\dim(\mathcal{H})$, then τ factorizes over the two factors of $\mathcal{H}$ so that τ is a separable state. Consider the segment with endpoints ρ and τ: $\rho_t = t \cdot \rho + (1-t) \cdot \tau, 0 \leqslant t \leqslant 1$. The modulus of separability ℓ [1] measures how far you can go towards ρ beginning at τ until you lose separability: $\ell(\rho) = \sup\{t : \rho_t \text{ is separable}\}$. The quantity $(1/\ell) - 1$ was studied by Vidal and Tarrach [4] as the 'random robustness of entanglement'. It can be shown [4, 1] that the supremum is a maximum, ρ_t is separable iff $t \leqslant \ell(\rho)$, $\ell(\rho) > 0$ and $1/\ell$ is a convex map on the states: for states ρ, ϕ and $0 \leqslant s \leqslant 1$,

$$\ell(s \cdot \rho + (1-s) \cdot \phi) \geqslant \left(\frac{s}{\ell(\rho)} + \frac{1-s}{\ell(\phi)}\right)^{-1}. \tag{1}$$

The other ingredient is the so-called gap representation of a state introduced in [1]. Let spec$(\rho) = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ and $\rho = \sum_{j=1}^{d} \lambda_j \cdot \rho_j$ be a spectral decomposition of ρ where ρ_j are pairwise orthogonal pure vector states. Define $\mu_j = j(\lambda_j - \lambda_{j+1}), j = 1, 2, \ldots, d-1$, and $\widehat{\rho}_j = j^{-1}\sum_{m=1}^{j} \rho_m, j = 1, 2, \ldots, d$. Note that $\sum_{j=1}^{d-1} \mu_j = 1 - d\lambda_d, \widehat{\rho}_d = \tau$ and

$$\text{spec}(\widehat{\rho}_j) = (\underbrace{1/j, 1/j, \ldots, 1/j}_{j}, 0, \ldots, 0) \equiv \mathrm{e}^{(j)}.$$

So $\widehat{\rho}_1$ is pure. Then a gap representation of ρ is $\rho = \sum_{j=1}^{d-1} \mu_j \cdot \widehat{\rho}_j + d\lambda_d \cdot \tau$. Noting that $\lambda_d = 1/d$ iff $\rho = \tau$, we assume that this is not the case and write

$$\rho = (1 - d\lambda_d) \cdot \omega + d\lambda_d \cdot \tau, \qquad \omega = \sum_{j=1}^{d-1} \frac{\mu_j}{1 - d\lambda_d} \cdot \widehat{\rho}_j.$$

By the results mentioned, ρ is separable iff

$$(1 - d\lambda_d) \leqslant \ell(\omega). \tag{2}$$

[1] We thank H Vogts and K Życzkowski for bringing [3] to our attention.

[2] A direct proof goes as follows. It suffices to show that if $\lambda \in \mathcal{B}$ with $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1/3$ then $\lambda \in \mathcal{C}$. Now putting $\lambda_4 = 1 - \lambda_1 - \lambda_2 - \lambda_3$ in the previous identity, we obtain

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - (\lambda_1 + \lambda_2 + \lambda_3) + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = -1/3.$$

Then, $(\lambda_1 - \lambda_3)^2 - 4\lambda_2\lambda_4 = \lambda_1^2 + \lambda_3^2 - 6\lambda_1\lambda_3 - 4\lambda_2 + 4\lambda_2^2 + 4(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)$. Using the displayed identity to eliminate the summand $4(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)$, we obtain $(\lambda_1 - \lambda_3)^2 - 4\lambda_2\lambda_4 = -3(\lambda_1 + \lambda_3 - 2/3)^2 \leqslant 0$, and this proves that $\lambda \in \mathcal{C}$.

Applying (1) to the state ω in its gap representation, we have

$$\ell(\omega) \geqslant \left(\sum_{j=1}^{d-1} \frac{\mu_j}{(1-d\lambda_d)\ell(\widehat{\rho}_j)} \right)^{-1} = (1-d\lambda_d) \left(\sum_{j=1}^{d-1} \frac{\mu_j}{\ell(\widehat{\rho}_j)} \right)^{-1},$$

thus (2) is satisfied (and thus ρ is separable) if $\sum_{j=1}^{d-1} \mu_j/\ell(\widehat{\rho}_j) \leqslant 1$. We can replace $\ell(\widehat{\rho}_j)$ by lower bounds.

Proposition 1. *If* $\ell(\widehat{\rho}_j) \geqslant p_j \geqslant 0$ *for* $j = 1, 2, \ldots, d-1$ *and* $\sum_{j=1}^{d-1} \mu_j/p_j \leqslant 1$*, then* ρ *is separable.*

The prime reason for introducing the gap representation is that not only the last summand τ but also the second last $\widehat{\rho}_{d-1}$ are separable. This follows from a result of Gurvits and Barnum [5]: if $\mathrm{tr}(\phi^2) \leqslant 1/(d-1)$ for a bipartite composite system of dimension d, then ϕ is separable. Now indeed $\mathrm{tr}(\widehat{\rho}_{d-1}^2) = 1/(d-1)$.

The least possible modulus of separability has been computed by Vidal and Tarrach [4]: $\inf\{\ell(\phi) : \phi \text{ a state}\} = 2/(2+d)$; the infimum is assumed at a pure state. To prove theorem 2, put $p_1 = p_2 = \cdots = p_{d-2} = 2/(2+d)$ and $p_{d-1} = 1$ in the proposition.

Turning to theorem 1, consider the numbers $\widehat{\ell}_j := \inf\{\ell(\phi) : \mathrm{spec}(\phi) = \mathrm{e}^{(j)}\}$, which give the minimal moduli of separability for the states spanning all possible gap representations. Replacing p_j by $\widehat{l}_j$ in the proposition gives us a general inequality providing a sufficient condition for separability. No general information is available for $\widehat{l}_j$ except the calculation of [6] for two qubits where $\widehat{l}_1 = 1/3, \widehat{l}_2 = 1/\sqrt{2}$ and $\widehat{l}_3 = 1$. From this and the proposition, one gets theorem 1. Since $\widehat{l}_1 = 1/3$ and $\widehat{l}_3 = 1$ follow from the results quoted above, we only give the calculation of $\widehat{l}_2$ in the appendix.

Appendix. Calculation of ℓ for a two-qubit state with spec = $(1/2, 1/2, 0, 0)$

Reference [6] gives a direct calculation of $\widehat{\ell}_1, \widehat{\ell}_2$ and $\widehat{\ell}_3$ using the Wootters criterion [7]. Recall that if ρ is a state of a two-qubit system, the Wootters operator W associated with it is $W = (\sqrt{\rho}(\sigma_y \otimes \sigma_y)\overline{\rho}(\sigma_y \otimes \sigma_y)\sqrt{\rho})^{1/2}$. Here all operators are taken as matrices with respect to a product orthonormal basis

$$\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

and $\overline{\rho}$ is the complex conjugate of ρ taken with respect to the basis which is real. The Wootters criterion is as follows: ρ is separable if and only if the (repeated) eigenvalues $w_1 \geqslant w_2 \geqslant w_3 \geqslant w_4$ of W satisfy $w_1 \leqslant w_2 + w_3 + w_4$.

We will calculate the modulus of separability for any state ρ for which $\mathrm{spec}(\rho) = (1/2, 1/2, 0, 0)$ by calculating the spectrum of the Wootters operator associated with $\rho_t = t\rho + (1-t)\tau, 0 \leqslant t \leqslant 1$. The spectrum of ρ_t consists of two double eigenvalues $\alpha = (1+t)/4$ and $\beta = (1-t)/4$ (which coincide for $t = 0$ where $\rho_0 = \tau$). In order not to overload the notation, we consider a density operator A with $\mathrm{spec}(A) = (\alpha, \alpha, \beta, \beta)$ where $\alpha + \beta = 1/2$ and $\alpha \geqslant \beta \geqslant 0$; thus $1/4 \leqslant \alpha \leqslant 1/2$. The spectral decomposition of A reads $A = \alpha P + \beta P^\perp$, where P is an orthoprojection of rank 2 and $P^\perp = \mathbf{1} - P$ is its orthocomplement, another orthoprojection of rank 2. It follows that $(\sigma_y \otimes \sigma_y)\overline{A}(\sigma_y \otimes \sigma_y) = \alpha Q + \beta Q^\perp$, where $Q = (\sigma_y \otimes \sigma_y)\overline{P}(\sigma_y \otimes \sigma_y)$ is an orthoprojection of rank 2 and $Q^\perp = \mathbf{1} - Q$. Using this one obtains for the square of the Wootters operator associated with A the formula $W^2 = \beta^2\mathbf{1} + \beta(\alpha-\beta)(P+Q) + (\alpha-\beta)(\sqrt{\alpha\beta}-\beta)(PQ+QP) + (\alpha^2-\beta^2-2\sqrt{\alpha\beta}(\alpha-\beta))PQP$.

Now since $P, P^\perp, Q$ and $Q^\perp$ are orthoprojections of rank 2 in a four-dimensional Hilbert space, we have three mutually exclusive alternatives for the subspaces U and V spanned by P and Q, respectively: (1) $U \cap V = \{0\}$ which happens when and only when $Q = \mathbf{1} - P$ which is equivalent to $\mathrm{tr}(PQ) = 0$; (2) $\dim(U \cap V) = 2$ which happens when and only when $Q = P$ which is equivalent to $\mathrm{tr}(PQ) = 2$; (3) $\dim(U \cap V) = 1$ which happens when and only when there are unit vectors ψ, ϕ and χ in the four-dimensional Hilbert space which satisfy $\langle\psi, \phi\rangle = \langle\psi, \chi\rangle = 0$ and $|\langle\chi, \phi\rangle| < 1$ such that $P = |\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|$ and $Q = |\psi\rangle\langle\psi| + |\chi\rangle\langle\chi|$. One has $\mathrm{tr}(PQ) = 1 + |\langle\chi, \phi\rangle|^2$. This alternative is equivalent to $\mathrm{tr}(PQ) \in [1, 2)$.

The three alternatives are distinguished by the value of $\mathrm{tr}(PQ)$. For convenience, we introduce the following characteristic geometric parameter $\xi = \mathrm{tr}(PQ) - 1$, which will determine the modulus of separability completely. We now distinguish the three possibilities:

(1) which occurs iff $\xi = -1$. Here $PQ = 0$ allows one to compute $W^2 = \alpha\beta\mathbf{1}$. The Wootters criterion is satisfied and the associated state is separable;

(2) which occurs iff $\xi = 1$. Here $P = Q$ allows one to calculate directly $W = A$, and the Wootters criterion is just $\alpha \leqslant 1/2$, so the associated state is separable;

(3) which occurs iff $0 \leqslant \xi < 1$. We may assume that

$$\psi = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \phi = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad \chi = \begin{pmatrix}0\\\sqrt{\xi}\\\eta_1\\\eta_2\end{pmatrix}, \qquad \eta = \begin{pmatrix}\eta_1\\\eta_2\end{pmatrix},$$

where $\eta \neq 0$ because $\|\eta\|^2 = \|\chi\|^2 - \xi = 1 - \xi > 0$. We now partition $\mathbb{C}^4 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^2$, and doing the necessary matrix multiplications we get, from our previous formula for W^2,

$$W^2 = \begin{pmatrix}\alpha^2 & 0 & \langle 0| \\ 0 & \alpha\beta + \alpha(\alpha - \beta)\xi & (\alpha - \beta)\sqrt{\xi\alpha\beta}\langle\eta| \\ |0\rangle & (\alpha - \beta)\sqrt{\xi\alpha\beta}|\eta\rangle & \beta^2\mathbf{1}_2 + \beta(\alpha - \beta)|\eta\rangle\langle\eta|\end{pmatrix}.$$

It is now clear that α^2 is an eigenvalue of W^2. The eigenvalue condition for an eigenvalue ζ to the eigenvector $x \oplus \mu$ for the lower right 3×3 block on $\mathbb{C} \oplus \mathbb{C}^2$ is

$$(\zeta - \alpha\beta - \alpha(\alpha - \beta)\xi)x = (\alpha - \beta)\sqrt{\xi\alpha\beta}\langle\eta, \mu\rangle, \tag{A.1}$$

$$(\zeta - \beta^2)\mu = (\alpha - \beta)(\sqrt{\xi\alpha\beta}x + \beta\langle\eta, \mu\rangle)\eta. \tag{A.2}$$

Putting $x = 0$ and taking as we may $\mu \neq 0$ orthogonal to η, (A.1) is satisfied and (A.2) reduces to $(\zeta - \beta^2)\mu = 0$; thus β^2 is an eigenvalue of W^2. We are now left with the problem of finding eigenvectors orthogonal to those already found. They are of the form $x \oplus c\eta$ with $x, c \in \mathbb{C}$. Inserting such eigenvectors into (A.1) and (A.2), the discussion of the solutions is tedious but straightforward. One obtains the two missing eigenvalues of W^2 to be $\zeta_\pm(\alpha, \xi) = \frac{\alpha}{2}(1 - 2\alpha) + \frac{\xi}{8}(4\alpha - 1)^2 \pm \frac{4\alpha - 1}{4}\sqrt{2\xi\alpha(1 - 2\alpha) + \xi^2(2\alpha - 1/2)^2}$. Having the four eigenvalues of A, we must decide which is the largest. We have $\alpha \geqslant \beta$ by assumption, and clearly $\zeta_+(\alpha, \xi) \geqslant \zeta_-(\alpha, \xi)$. Moreover, $\xi \mapsto \zeta_+(\alpha, \xi)$ is increasing and $\zeta_+(\alpha, 1) = \alpha^2$. Thus, α is the largest eigenvalue of W and the Wootters criterion reads $\alpha \leqslant \beta + \sqrt{\zeta_+(\alpha, \xi)} + \sqrt{\zeta_-(\alpha, \xi)}$. Manipulation of this inequality shows that it is equivalent to $\alpha \leqslant (1 + (1/\sqrt{2 - \xi}))/4$.

Recalling that $\alpha = (1 + t)/4$, we arrive at the following: if the two-qubit state ρ has $\mathrm{spec}(\rho) = (1/2, 1/2, 0, 0)$, then

$$\ell(\rho) = \begin{cases} 1, & \text{if } Q = \mathbf{1} - P, \\ \frac{1}{\sqrt{3 - \mathrm{tr}(PQ)}}, & \text{otherwise,} \end{cases}$$

where P is the spectral orthoprojection to the eigenvalue 1/2 and $Q = (\sigma_y \otimes \sigma_y)\overline{P}(\sigma_y \otimes \sigma_y)$. Since $\mathrm{tr}(PQ) \in [1, 2]$ when $Q \neq \mathbf{1} - P$, we obtain $\widehat{\ell_2} = \inf\{\ell(\rho) : \mathrm{spec}(\rho) = (1/2, 1/2, 0, 0)\} = 1/\sqrt{2}$.

References

[1] Raggio G A 2006 Spectral conditions on the state of a composite quantum system implying its separability *J. Phys. A: Math. Gen.* **39** 617

[2] Życzkowski K, Horodecki P, Sanpera A and Lewenstein M 1998 Volume of the set of separable states *Phys. Rev.* A **58** 883

[3] Verstraete F, Audenaert K and De Moor B 2001 Maximally entangled mixed states of two qubits *Phys. Rev.* A **64** 012316

[4] Vidal G and Tarrach R 1999 Robustness of entanglement *Phys. Rev.* A **59** 141

[5] Gurvits L and Barnum H 2003 Separable balls around the maximally mixed multipartite quantum state *Phys. Rev.* A **68** 042312

[6] Gabach Clément M E 2005 Entrelazamiento en sistemas cuánticos compuestos *Trabajo Especial de Licenciatura en Física* (FaMAF), unpublished

[7] Wootters W K 1998 Entanglement of formation of an arbitrary state of two qubits *Phys. Rev. Lett.* **80** 2245